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A statistical law for multiplicities of $SU(3)$ irreps (λ, μ) in the plethysm $\{\eta\} \overset{3}{\otimes} \{m\} \to (\lambda, \mu)$

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Abstract

A statistical law for the multiplicities of the $SU(3)$ irreps (λ, μ) in the reduction of totally symmetric irreducible representations $\{m\}$ of $U(\mathcal{N}), \mathcal{N} = (\eta + 1)(\eta + 2)/2$ with η being the three-dimensional oscillator major shell quantum number, is derived in terms of the quadratic and cubic invariants of $SU(3)$, by determining the first three terms of an asymptotic expansion for the multiplicities. To this end, the bivariate Edgeworth expansion known in statistics is used. Simple formulae, in terms of m and η, for all the parameters in the expansion are derived. Numerical tests with large m and $\eta = 4, 5$ and 6 show good agreement with the statistical formula for the $SU(3)$ multiplicities.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

A statistical law for the multiplicities $D(m, L)$ of the $SO(3)$ irreps $[L]$ in the reduction of totally symmetric irreducible representations (irreps) $\{m\}$ of $U(\mathcal{N})$ in $U(\mathcal{N}) \to SO(3)$, with the basic association $\{1\}_{U(\mathcal{N})} \to \sum_i [\ell_i] \oplus$, i.e. number of times the irrep L appears in $\{m\} \to L$, is well known [1, 2]. This has been derived both by using the 'plethysm' formulation of the problem [3] and by using the $SO(2)$ subgroup of $SO(3)$ [1, 4]. Appendix A gives some details of the second method as its extension is used in the present paper. The statistical law for $D(m, L)$, involving only the dimension $(2L + 1)$ and the L^2 eigenvalue $L(L + 1)$, allows one to estimate dimensions of Hamiltonian matrices in interacting boson models (IBMs) of atomic nuclei [1, 2]. More importantly, by including energy dependence in the parameters of $D(m, L)$, this gives angular momentum decomposition of level densities, an application of great interest in nuclear physics [5–7]. As emphasized by Wybourne, deriving statistical laws for multiplicities in the reduction of an irrep of G into irreps of K in $G \supset K$ belong to statistical group theory [3], a subject that is not well developed yet. Going beyond $SO(3)$ multiplicities, in the past [6, 8] there were some attempts to derive statistical laws involving $SU(3)$ and $SU(4)$ algebras that appear in nuclear physics. However the formulae derived are incomplete as discussed in detail below for $SU(3)$.

Our purpose in this paper is to take a step forward in the subject of 'statistical group theory' and derive the statistical law for the multiplicities of the $SU(3)$ irreps (λ, μ) in the reduction of a totally symmetric irrep $\{m\}$ of $U(\mathcal{N})$, $\mathcal{N} = (\eta+1)(\eta+2)/2$. This is equivalent to developing a statistical theory for the plethysm

$$\{\eta\} \overset{3}{\otimes} \{m\} \rightarrow \sum_{(\lambda,\mu)} D^{(\lambda,\mu)}_{\{\eta\},\{m\}}(\lambda, \mu), \qquad \text{with} \qquad \{1\}_{U(\mathcal{N})} \rightarrow (\eta, 0)_{SU(3)}. \tag{1}$$

The symbol $\otimes$ denotes (complete) plethysm while $\overset{3}{\otimes}$ denotes a plethysm in which only the Schur (S) functions with no more than three rows are considered. Note that given a $U(3)$ irrep $\{f_1, f_2, f_3\}$, the corresponding $SU(3)$ irrep is (λ, μ) with $\lambda = f_1 - f_2$ and $\mu = f_2 - f_3$. Equation (1) is relevant in physical applications [1, 2] since $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ gives the number of times that a given irrep (λ, μ) of $SU(3)$ occurs when we distribute m bosons into the states of irrep $\{\eta\}$ of $SU(\mathcal{N})$; here η denotes 3D oscillator major shell number and $\eta = 2(\mathcal{N} = 6)$ for sd IBM and $\eta = 4(\mathcal{N} = 15)$ for sdg IBM [9]. Also, as $SU(3)$ represents deformed nuclei [9], the $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ will be useful in incorporating deformation effects in nuclear level densities [8].

For $\eta = 2$ one can use the result, due to Littlewood [10], that the (complete) plethysm $\{2\}\otimes\{m\}$ consists of all Schur functions of degree $2m$ with even entries and multiplicities 1. From this result it follows that

$$\{2\} \overset{3}{\otimes} \{m\} = \sum (2(f_1 - f_2), 2(f_2 - f_3)), \qquad \text{with} \qquad f_1 + f_2 + f_3 = m. \tag{2}$$

For $\eta > 2$ the use of formulae given in section 2 allows one to compute $\{\eta\} \overset{3}{\otimes} \{m\}$ for any η and m. From these calculations one realizes that the number of multiplets (λ, μ) as well as their multiplicities $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ grow very fast with increasing η and m. For example, for $\eta = 4$ and $m = 20$ one has 566 multiplets with multiplicities as big as 3148 while for $\eta = 5$ and $m = 20$ those numbers are 879 and 187 328. This justifies the search for a statistical law for equation (1). Now we will give a preview.

In section 2, a summary of two exact methods for solving equation (1) is given. Also discussed here are trivial zeros of $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$. In section 3, as a starting point for developing a statistical theory, $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ is expressed as a trace over the m particle spaces involving the quadratic and cubic invariant of $SU(3)$ and this makes clear that D can be interpreted as a bivariate density. An exact method (this is a third method besides the two methods discussed in section 2) based on $U(3) \supset U(1) \oplus U(1) \oplus U(1)$ chain allows us to derive asymptotic expansions for the $SU(3)$ multiplicities. This method, employed first by Kanestrom [8], is formulated in detail in section 4. Here the joint distribution of the eigenvalues of $\gamma = (N_z - N_x)$ and $\nu = (N_x - N_y)$ for m bosons in an oscillator shell η (with N_i being number of quanta in the ith direction) is considered and propagation equations are derived for the lower order bivariate moments of this distribution. Approximating the distribution by a bivariate Gaussian gives a statistical formula for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$. In section 5, formulae for the first two corrections are derived and they result in a good asymptotic expression for Ds involving both the quadratic and cubic invariants of $SU(3)$. This is the main result of the paper. Some numerical tests of the statistical formula are also given in section 5. Finally section 6 gives conclusions.

2. Plethysm method for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$

A general expression for plethysm of S-functions, by using equation (40) of [11], is

$$\{\eta\}\otimes\{f\} = [r!]^{-1}\sum_k h_k C_k^{\{f\}}(\{\eta\}\otimes S_1)^{\alpha}(\{\eta\}\otimes S_2)^{\beta}\cdots. \tag{3}$$

In equation (3), $\{f\}$ is a partition (i.e. Young tableaux) of the integer r, k is a class of the symmetric group of order $r!$ and specified by the cyclic structure $(1^{\alpha}, 2^{\beta}, \ldots)$, S_i is a symmetric function (power sum; see [11]), h_k is the order of the class k and $C_k^{\{f\}}$ is the character of the class k corresponding to the partition $\{f\}$. The general result for $\{\eta\}\otimes S_r$ restricted to irreps with maximum three rows is [12]

$$\begin{aligned}\{\eta\} \overset{3}{\otimes} S_p = \sum_{a,b} [&\{\eta p - ap, ap - bp, bp\} - \{\eta p - ap, ap - bp - 1, bp + 1\}\\ &+ \{\eta p - ap - 1, ap - bp - 1, bp + 2\} - \{\eta p - ap - 1, ap - bp + 1, bp\}\\ &+ \{\eta p - ap - 2, ap - bp + 1, bp + 1\} - \{\eta p - ap - 2, ap - bp, bp + 2\}]\,.\end{aligned} \tag{4}$$

In (4) the summation is over all positive integers a and b with the constraint that all the non-standard $\{f_1, f_2, f_3\}$ irreps to be ignored. With $r = m$ and $\{f\} = \{m\}$, equations (3) and (4) solve $\{m\}_{U(\mathcal{N})} \to (\lambda\mu)_{SU(3)}$ by using the fact that $C_k^{\{m\}} = 1$ independent of k. This method was used to get the Ds for $\eta = 4$, $m \leqslant 15$ in the past [4] and also for some examples with $\eta = 5$ [1]. It is useful to note that equations (3) and (4) can be used easily for antisymmetric irreps $\{f\} = \{1^m\}$ as $C_k^{\{1^m\}} = \pm 1$. Combining these with the result for expanding any S-function into Kronecker products of symmetric or antisymmetric S-functions will generate the reductions for any irrep of $U(\mathcal{N})$ into $SU(3)$ irreps.

Another method for the exact calculation of the plethysm in equation (1) is given in [13] and it uses the recursion formula [14]

$$\{\eta\}\otimes\{m\} = \frac{1}{m}\sum_{k=1}^{m}\left[\sum_{\nu} C_{\eta,k,\{\nu\}}\{\nu\}\right](\{\eta\}\otimes\{m-k\}); \qquad m \geqslant 2 \tag{5}$$

with initial input $\{\eta\}\otimes\{1\} = \{\eta\}$. The sum involving $\{\nu\}$ includes all partitions of ηk and the coefficients $C_{\eta,k,\{\nu\}}$ have values $0, +1, -1$. The value of $C_{\eta,k,\{\nu\}}$ is obtained removing, in sequence, from the Young diagram associated with $\{\nu\}$, ηk-border strips. If in all steps the resulting diagram represents a standard partition then

$$C_{\eta,k,\{\nu\}} = (-1)^{\ell} \tag{6}$$

with $\ell =$ (number of lines in the removed k-border strips)$-\eta$. If in some step the resulting diagram does not represent a standard partition then $C_{\eta,k,\{\nu\}} = 0$. A k-border strip of a Young diagram of a partition $\{\nu\}$ is a sequence of k squares in which the first of them is the last one of the first line of $\{\nu\}$ and the next square to a given one is the one below it, if it exists, or the one to its left, otherwise. If in all the steps of the recurrent process to obtain $\{\eta\}\otimes\{m\}$ given by equation (5) one considers only the partitions $\{\nu\}$ with no more than p rows, one obtains a p-reduced plethysm [15].

Using the exact methods, tabulations are generated for $\eta = 4$ with $m = 1$–48, for $\eta = 5$ with $m = 1$–26 and for $\eta = 6$ with $m = 1$–19. These are available upon request. Some of these results are shown in figures 1–4. It is clear from these figures that the multiplicities are very large for large m and hence the need for statistical laws. In order to have an insight on the dependence of $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ on λ and μ for given values of η and m, we used the results of exact calculations for $\{\eta\} \overset{3}{\otimes} \{m\}$ to make a two-dimensional tabulation of $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ with λ in the rows and μ in the columns for $\eta = 4$ and 5 and m from 10 to 20. In these tabulations appear sequences of zeros in straight lines of type $\lambda = \mu + X$ and in the upper right triangular region. These zeros are not a peculiarity of $\{\eta\} \overset{3}{\otimes} \{m\}$ but only a consequence of the definitions $\lambda = f_1 - f_2$, $\mu = f_2 - f_3$ and $\eta m = f_1 + f_2 + f_3$ where $\{f_1, f_2, f_3\}$ are the $U(3)$ irreps that appear in the expansion of $\{\eta\} \overset{3}{\otimes} \{m\}$. In fact, from these conditions, one obtains

$$\begin{aligned} \lambda &= \mu + 3(f_1 + f_3) - 2\eta m, \\ \lambda + 2\mu &= \eta m - 3f_3. \end{aligned} \tag{7}$$

These equations imply that

$$D^{(\lambda,\mu)}_{\{\eta\},\{m\}} = 0 \qquad \text{for} \qquad \begin{cases} \mu \neq (\lambda + 2\eta m)(\text{mod } 3) \\ \lambda + 2\mu > \eta m. \end{cases} \tag{8}$$

Also note that $2\lambda + \mu = (2\eta m) - 3k$ with $k = 0, 1, \ldots, \left[\frac{2\eta m}{3}\right]$ and $\mu = k, k-2, \ldots, 0$ or 1. There are other non-trivial zeros due to selection rules. For example for the symmetric irrep $\{m\}$, the $SU(3)$ irrep $(\eta m - 2, 1)$ is not allowed.

3. $SU(3)$ multiplicities in trace form involving quadratic and cubic Casimir invariants

Let us begin with a brief discussion of the generators and Casimir invariants of $SU(3)$. The Lie algebra of $U(3)$ has the generators $\mathcal{C}_i^j$ satisfying the commutation relations

$$\left[\mathcal{C}_i^j, \mathcal{C}_k^\ell\right] = \mathcal{C}_i^\ell \delta_k^j - \mathcal{C}_k^j \delta_i^\ell; \qquad i, j, k, \ell = 1, 2, 3. \tag{9}$$

The generators, with definite tensorial rank with respect to $SO(3)$ in $SU(3) \supset SO(3)$, of the Lie algebra of $SU(3)$ are

$$\begin{aligned} &L_{+1} = -\mathcal{C}_1^2 - \mathcal{C}_2^3, \qquad L_{-1} = \mathcal{C}_2^1 + \mathcal{C}_3^2, \qquad L_0 = \mathcal{C}_1^1 - \mathcal{C}_3^3, \\ &Q_2 = \sqrt{6}\mathcal{C}_1^3, \qquad Q_1 = \sqrt{3}\left(\mathcal{C}_2^3 - \mathcal{C}_1^2\right), \qquad Q_0 = \mathcal{C}_1^1 - 2\mathcal{C}_2^2 + \mathcal{C}_3^3, \\ &Q_{-2} = \sqrt{6}\mathcal{C}_3^1, \qquad Q_{-1} = \sqrt{3}\left(\mathcal{C}_2^1 - \mathcal{C}_3^2\right). \end{aligned} \tag{10}$$

Here we are following [16] with convenient overall factors as to make $L_q (q = -1, 0, +1)$ and $Q_q (q = \pm 1, \pm 2, 0)$ exact components of $SO(3)$ Racah tensors [17] of ranks 1 and 2. The commutation relations for Ls and Qs are (see for example equation (117) in [1])

$$\begin{aligned} [L_q, L_{q'}] &= -\sqrt{2}\langle 1q1q'|1q+q'\rangle L_{q+q'}, \\ [Q_q, Q_{q'}] &= 3\sqrt{10}\langle 2q2q'|1q+q'\rangle L_{q+q'}, \\ [Q_q, L_{q'}] &= -\sqrt{6}\langle 2q1q'|2q+q'\rangle Q_{q+q'}. \end{aligned} \tag{11}$$

The second-order $SO(3)$ scalars that can be constructed using L_q and Q_q operators are

$$\begin{aligned} L^2 &= -2L_{-1}L_{+1} + L_0(L_0 + 1), \\ [Q, Q]^0_0 &= \frac{1}{\sqrt{5}}(2Q_{-2}Q_2 - 2Q_{-1}Q_1 + Q_0Q_0 + 9L_0). \end{aligned} \tag{12}$$

With these second-order scalars one can construct the $SU(3)$ quadratic Casimir invariant $\mathcal{C}_2(SU(3))$,

$$\mathcal{C}_2(SU(3)) = \tfrac{1}{4}\left(3L^2 + \sqrt{5}[Q, Q]^0_0\right) \tag{13}$$

with eigenvalues

$$\langle \mathcal{C}_2(SU(3))\rangle^{(\lambda,\mu)} = C_2(\lambda, \mu) = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu). \tag{14}$$

Similarly the third-order $SU(3)$ Casimir invariant $\mathcal{C}_3(SU(3))$ is

$$\begin{aligned}\mathcal{C}_3(SU(3)) &= \frac{1}{72}\left(\sqrt{70}[[Q,Q]^2,Q]^0_0 + 9\sqrt{30}[[L,L]^2,Q]^0_0\right)\\ &= \frac{1}{72}[12Q_{-2}Q_0Q_2 - 3\sqrt{6}Q_{-2}Q_1Q_1 - 3\sqrt{6}Q_{-1}Q_{-1}Q_2 + 6Q_{-1}Q_0Q_1\\ &\quad - 2Q_0Q_0Q_0 - 27\sqrt{3}L_{-1}Q_1 - 27\sqrt{3}Q_{-1}L_{+1} + 27L_0Q_0 + 45Q_0\\ &\quad + 9(\sqrt{6}Q_{-2}L_{+1}L_{+1} - 2\sqrt{3}Q_{-1}L_0L_{+1} + 2L_{-1}Q_0L_{+1} + 2L_0L_0Q_0\\ &\quad - 2\sqrt{3}L_{-1}L_0Q_1 + \sqrt{6}L_{-1}L_{-1}Q_2 - \sqrt{3}Q_{-1}L_{+1} + 5L_0Q_0\\ &\quad - \sqrt{3}L_{-1}Q_1 + 3Q_0)]\end{aligned} \tag{15}$$

with eigenvalues

$$\langle\mathcal{C}_3(SU(3))\rangle^{(\lambda,\mu)} = C_3(\lambda,\mu) = (1/9)(\lambda-\mu)(\lambda+2\mu+3)(2\lambda+\mu+3) \tag{16}$$

as given by Draayer and Rosensteel [18]. Note that $C_2(\lambda,\mu) = C_2(\mu,\lambda)$ but this is not true for $C_3(\lambda,\mu)$. Finally let us give the formula for the dimension $d(\lambda,\mu)$ of the $SU(3)$ irrep (λ,μ),

$$d(\lambda,\mu) = (\lambda+1)(\mu+1)(\lambda+\mu+2)/2. \tag{17}$$

We will now derive the trace form for the Ds.

With $\beta_{(\lambda,\mu)}$ labeling the multiple occurrence of a given (λ,μ) irrep in $\{m\}$ and $\alpha_{(\lambda,\mu)}$ labeling the different states that belong to a given (λ,μ), we have

$$\begin{aligned}&[d(\lambda,\mu)]^{-1}\langle\langle\delta(\mathcal{C}_2(SU(3)) - C_2(\lambda,\mu))\delta(\mathcal{C}_3(SU(3)) - C_3(\lambda,\mu))\rangle\rangle^{\{m\}}\\ &\quad = [d(\lambda,\mu)]^{-1}\sum_{\beta_{(\lambda',\mu')},(\lambda',\mu'),\alpha_{(\lambda',\mu')}}\langle\{m\},\beta_{(\lambda',\mu')},(\lambda',\mu'),\alpha_{(\lambda',\mu')}|\\ &\delta(\mathcal{C}_2(SU(3)) - C_2(\lambda,\mu))\delta(\mathcal{C}_3(SU(3)) - C_3(\lambda,\mu))|\{m\},\beta_{(\lambda',\mu')},(\lambda',\mu'),\alpha_{(\lambda',\mu')}\rangle\\ &\quad = [d(\lambda,\mu)]^{-1}\sum_{\beta_{(\lambda',\mu')},(\lambda',\mu'),\alpha_{(\lambda',\mu')}}\langle\{m\},\beta_{(\lambda',\mu')},(\lambda',\mu'),\alpha_{(\lambda',\mu')}|\\ &\delta(C_2(\lambda',\mu') - C_2(\lambda,\mu))\delta(C_3(\lambda',\mu') - C_3(\lambda,\mu))|\{m\},\beta_{(\lambda',\mu')},(\lambda',\mu'),\alpha_{(\lambda',\mu')}\rangle\\ &\quad = [d(\lambda,\mu)]^{-1}\sum_{\beta_{(\lambda,\mu)},\alpha_{(\lambda,\mu)}} 1\\ &\quad = D^{(\lambda,\mu)}_{\{\eta\},\{m\}}.\end{aligned} \tag{18}$$

Note that $\langle\langle\rangle\rangle$ denotes trace (sum of diagonal matrix elements in a given basis). In the first step we have used, as the trace is invariant under a basis transformation, the basis defined by $SU(3)$ irreps. In the next step, we have used the action of $SU(3)$ Casimir operators on a $SU(3)$ irrep. Finally the definition of the delta function ($\delta(x) = 1$ for $x = 0$ and zero otherwise) is used. With these, the final result is obtained. Thus $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ can be expressed as a trace over the m boson spaces. If the $\mathcal{C}_3(SU(3))$ is dropped, then the β summation at the end will involve both (λ,μ) and (μ,λ) irreps and hence it is not possible to define Ds using only $\mathcal{C}_2(SU(3))$. Note that we have the equality

$$\sum_{(\lambda,\mu)} D^{(\lambda,\mu)}_{\{\eta\},\{m\}} d(\lambda,\mu) = d(\mathcal{N},m) = \binom{\mathcal{N}+m-1}{m}. \tag{19}$$

More importantly, it is seen that the trace in equation (18) is positive definite and hence it can be treated as a bivariate probability density $\rho'(E_1,E_2)$ in the two variables E_1 and E_2, with E_1s being the eigenvalues of $\mathcal{C}_2(SU(3))$ and E_2s being the eigenvalues of $\mathcal{C}_3(SU(3))$. Then the bivariate moments M_{PQ} of $\rho'(E_1,E_2)$ are given by $M_{PQ} =$

$[d(\lambda,\mu)]^{-1}\langle\langle[\mathcal{C}_2(SU(3))]^P[\mathcal{C}_3(SU(3))]^Q\rangle\rangle^{\{m\}}$. Our task is to identify a functional form for this bivariate probability density or more directly for $\rho(\lambda,\mu)$ (note that $\mathrm{d}E_1\,\mathrm{d}E_2 \neq \mathrm{d}\lambda\,\mathrm{d}\mu$) and hence a statistical law for the Ds. The remaining part of the paper deals with $\rho(\lambda,\mu)$.

4. A statistical law for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ based on $U(3) \supset U(1) \oplus U(1) \oplus U(1)$

4.1. Preliminaries

Extending the formulation based on $SO(3) \supset SO(2)$ for $D(m,L)$ as discussed in appendix A, it is possible to derive a statistical law for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$. To this end, for $U(\mathcal{N})$ with $\mathcal{N} = (\eta+1)(\eta+2)/2$, we consider the oscillator single particle (sp) states in the (n_z, n_x, n_y) representation, i.e. we consider $(n_z(i), n_x(i), n_y(i))$ orbits with $i = 1, 2, \ldots, \mathcal{N}$. Here $n_z(i), n_x(i)$ and $n_y(i)$ are the number of oscillator quanta for a single particle in z, x and y directions respectively; $n_x(i) + n_y(i) + n_z(i) = \eta$ and n_z, n_x and n_y are positive. For example for $\eta = 4$ we have $\mathcal{N} = 15$ and (n_z, n_x, n_y) = (400), (310), (301), (220), (211), (202), (130), (121), (112), (103), (040), (031), (022), (013) and (004). Now, distributing m bosons in these sp states in all possible ways gives the number of quanta $f_1 = N_z$, $f_2 = N_x$ and $f_3 = N_y$ in z, x and y directions for each distribution (or configuration). Note that $f_1 = N_z = \sum_{i=1}^{\mathcal{N}} m_i n_z(i)$, where m_i is the number of bosons in the i-th sp state. Similarly f_2 (or N_x) and f_3 (or N_y) can be obtained; $f_1+f_2+f_3 = m\eta$. It is possible to generate all allowed configurations using a program and count the number of configurations giving the same (f_1, f_2, f_3). Let us call this function $d(f_1, f_2, f_3)$. We have written a programme to generate $d(f_1, f_2, f_3)$.

For each $SU(3)$ irrep (λ,μ), there are a set of weights (f_1, f_2, f_3) (they are the subgroup labels in $U(3) \supset U(1) \oplus U(1) \oplus U(1)$) that belong to this irrep with the highest one denoted by (F_1, F_2, F_3) such that $\lambda = F_1 - F_2$, $\mu = F_2 - F_3$ and $F_1 + F_2 + F_3 = m\eta$. Introducing the operators $\mathcal{O}_i$ which increase the variable f_i by one unit so that for example $\mathcal{O}_1^p g(f_1, f_2, f_3) = g(f_1 + p, f_2, f_3)$ etc, number of times (λ,μ) irrep appears in $\{m\}$ is given by [6, 8]

$$D^{(\lambda,\mu)}_{\{\eta\},\{m\}} = \begin{vmatrix} 1 & \mathcal{O}_2^{-1} & \mathcal{O}_3^{-2} \\ \mathcal{O}_1^{1} & 1 & \mathcal{O}_3^{-1} \\ \mathcal{O}_1^{2} & \mathcal{O}_2^{1} & 1 \end{vmatrix} d(F_1, F_2, F_3). \tag{20}$$

Equation (20) is exact and it is an extension of equation (A.6). We have verified equation (20) explicitly for $\eta = 4$ and $m = 10, 15, 20$ using a computer programme we have developed. Thus equation (20) provides an exact method for generating $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$. However, unlike the two methods discussed in section 2, equation (20) allows us to derive smooth functional forms for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$. The clue lies in approximating the function $d(F_1, F_2, F_3)$. To this end, with $\gamma = f_1 - f_2$ and $\nu = f_2 - f_3$ we can first recognize that $\rho_d(\gamma,\nu) = d(f_1, f_2, f_3)/d(\mathcal{N}, m)$ can be treated as a bivariate probability distribution (so also is $\rho_D(\lambda,\mu) = \frac{\mathrm{d}(\lambda,\mu)}{\mathrm{d}(\mathcal{N},m)} D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$). Given the operators $\hat{N}_z$, $\hat{N}_x$ and $\hat{N}_y$ that generate m particle N_z, N_y and N_x values, the operators generating the variables γ and ν are $\hat{\gamma} = \hat{N}_z - \hat{N}_x$ and $\hat{\nu} = \hat{N}_x - \hat{N}_y$. Clearly, $\hat{\gamma}$ and $\hat{\nu}$ are one-body operators. Therefore, following the results in [19] (see also appendix A), it is seen that the marginal densities of $\rho_d(\gamma,\nu)$ for a m boson system with large m value will be in general close to Gaussian. Therefore a good starting point is to approximate $\rho_d(\gamma,\nu)$ by a bivariate Gaussian $\rho_{d:\mathcal{G}}$. Then we need the two centroids, two variances and the correlation coefficient defining $\rho_{d:\mathcal{G}}$. Derivations of the formulae for these are as follows.

4.2. Propagation equations for the second-order bivariate moments of $\rho_d(\gamma, \nu)$

Starting with the definitions $\hat{\gamma} = \hat{N}_z - \hat{N}_x$ and $\hat{\nu} = \hat{N}_x - \hat{N}_y$ and applying the symmetries of $\hat{N}_i$ operators and the trace propagation equations given by equations (A.2)–(A.5), we have

$$\langle\hat{\gamma}\rangle^m = m\langle\hat{\gamma}\rangle^1 = m\langle\hat{N}_z - \hat{N}_x\rangle^1 = 0, \qquad \langle\hat{\nu}\rangle^m = m\langle\hat{\nu}\rangle^1 = 0. \tag{21}$$

Therefore the operators $\hat{\gamma}$ and $\hat{\nu}$ are traceless operators (i.e. their m particle centroids are zero) and then the marginal variances are

$$\begin{aligned}\sigma_\gamma^2(m) &= \langle\hat{\gamma}^2\rangle^m = \langle(\hat{N}_z - \hat{N}_x)^2\rangle^m = 2\left[\left\langle\widetilde{N}_z^2\right\rangle^m - \langle\widetilde{N}_z\widetilde{N}_x\rangle^m\right],\\ \sigma_\nu^2(m) &= \langle\hat{\nu}^2\rangle^m = \sigma_\gamma^2(m).\end{aligned} \tag{22}$$

In equation (22) $\widetilde{N}_i$ are traceless $\hat{N}_i$ operators, i.e. $\langle\widetilde{N}_i\rangle^m = 0$. Also used here are the symmetries of $\hat{N}_i$ operators. In the discussion ahead we will also use the result that $\hat{N}_i$ and $\hat{N}_j$ commute for any (i, j). Now let us consider the bivariate correlation coefficient ζ defined by

$$\begin{aligned}\zeta(m) &= \frac{\langle\hat{\gamma}\,\hat{\nu}\rangle^m}{\sigma_\gamma^2(m)},\\ \langle\hat{\gamma}\,\hat{\nu}\rangle^m &= \langle(\hat{N}_z - \hat{N}_x)(\hat{N}_x - \hat{N}_y)\rangle^m\\ &= \langle\widetilde{N}_z\widetilde{N}_x\rangle^m - \langle\widetilde{N}_z^2\rangle^m.\end{aligned} \tag{23}$$

Here we have used the identities $\langle\widetilde{N}_z\widetilde{N}_x\rangle^m = \langle\widetilde{N}_z\widetilde{N}_y\rangle^m = \langle\widetilde{N}_x\widetilde{N}_y\rangle^m$ and $\left\langle\widetilde{N}_z^2\right\rangle^m = \left\langle\widetilde{N}_x^2\right\rangle^m$. Comparing equations (22) and (23) gives the important result,

$$\zeta(m) = -\tfrac{1}{2}. \tag{24}$$

To derive the propagation formula for the marginal variances $\sigma_\gamma^2(m)$, we need $\left\langle\widetilde{N}_z^2\right\rangle^m$ and $\langle\widetilde{N}_z\widetilde{N}_x\rangle^m$ for any m. Propagation equations for these are (see appendix A)

$$\begin{aligned}\left\langle\widetilde{N}_z^2\right\rangle^m &= \frac{m(m+\mathcal{N})}{\mathcal{N}(\mathcal{N}+1)}\left\langle\!\left\langle\widetilde{N}_z^2\right\rangle\!\right\rangle^1,\\ \langle\widetilde{N}_z\widetilde{N}_x\rangle^m &= \frac{m(m+\mathcal{N})}{\mathcal{N}(\mathcal{N}+1)}\langle\langle\widetilde{N}_z\widetilde{N}_x\rangle\rangle^1.\end{aligned} \tag{25}$$

The one particle traces in equation (25) are obtained as follows. Firstly,

$$\langle\langle\hat{N}_z\rangle\rangle^1 = \sum_{i=1}^{\eta+1}(\eta - i + 1)(i) = \mathcal{N}\left(\frac{\eta}{3}\right). \tag{26}$$

Therefore $\langle\hat{N}_z\rangle^1 = \langle\hat{N}_x\rangle^1 = \langle\hat{N}_y\rangle^1 = \eta/3$. Note that $(\eta - i + 1)$ are the eigenvalues of $\hat{N}_z$ and i is the degeneracy of the ith eigenvalue. With this the trace of the square of $\widetilde{N}_z$ over one particle spaces is

$$\begin{aligned}\left\langle\!\left\langle\widetilde{N}_z^2\right\rangle\!\right\rangle^1 &= \sum_{i=1}^{\eta+1}\left(\frac{2\eta}{3} + 1 - i\right)^2(i)\\ &= \frac{\eta(\eta+1)(\eta+2)(\eta+3)}{36}.\end{aligned} \tag{27}$$

Now we will turn to $\langle\langle\widetilde{N}_z\widetilde{N}_x\rangle\rangle^1$. For this, first note that for a given n_z, say $\eta - r$, the n_x takes values $0, 1, 2, \ldots, r$. Therefore

$$\langle\langle \widetilde{N}_z \widetilde{N}_x \rangle\rangle^1 = \sum_{i=0}^{\eta} \left(\frac{2\eta}{3} - i \right) \left[\sum_{j=0}^{i} \left(j - \frac{\eta}{3} \right) \right] = \frac{1}{18} \sum_{i=0}^{\eta} (2\eta - 3i)(i+1)(3i - 2\eta) = -\frac{\eta(\eta+1)(\eta+2)(\eta+3)}{72}. \tag{28}$$

Combining equations (27) and (28) we finally have

$$\sigma_\gamma^2(m) = \frac{m(m+\mathcal{N})}{\mathcal{N}(\mathcal{N}+1)} \left[\frac{\eta(\eta+1)(\eta+2)(\eta+3)}{12} \right]. \tag{29}$$

For example, for $\eta = 4$ and $m = 10, 15, 20$ the $\sigma_\gamma^2(m)$ values are $875/12$, $525/4$ and $1225/6$ respectively. These numbers are verified by numerically generating $d(\gamma, \nu)$.

With the marginal centroids zero, marginal variances given by equation (29) and the correlation coefficient $\zeta(m)$ being $-\frac{1}{2}$, the bivariate Gaussian in γ and ν is

$$\rho_{d;\mathcal{G}}(\gamma, \nu) = \frac{N_0}{\sqrt{3}\pi\sigma_\gamma^2(m)} \exp\left[-\frac{2}{3\sigma_\gamma^2(m)} (\gamma^2 + \gamma\nu + \nu^2) \right]. \tag{30}$$

The factor $N_0 = 3$ in equation (30) and this is due to the following. As $\gamma = N_z - N_x$ and $\nu = N_x - N_y$, we have $\nu = (\gamma - \eta m) + 3N_x$. Therefore as γ increases in steps of one, ν will change in steps of three. Thus, when we use only the allowed values of (γ, ν) in applying equation (30), we need the factor $N_0 = 3$ for proper normalization. As we shall see ahead in section 5, although equation (30) is a good starting point, we do need first two corrections to this formula.

4.3. Formula for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ from the bivariate Gaussian form for $\rho_{d;\mathcal{G}}(\gamma, \nu)$

Starting with the bivariate Gaussian form given by equation (30) and applying equation (20), we can derive a smooth formula for $\rho_D(\lambda, \mu)$. To this end we will follow [6] and replace the difference operators O_i^n by differential operators using the Taylor expansion,

$$O_i^n = 1 + n\frac{\partial}{\partial F_1} + \frac{n^2}{2!}\left(\frac{\partial}{\partial F_1}\right)^2 + \cdots. \tag{31}$$

For better accuracy, the determinant in equation (20) is transformed so that the arguments in the Taylor series become as small as possible. Toward this end we use (see for example [6])

$$\begin{vmatrix} 1 & O_2^{-1} & O_3^{-2} \\ O_1^1 & 1 & O_3^{-1} \\ O_1^2 & O_2^1 & 1 \end{vmatrix} = O_1^1 O_3^{-1} \begin{vmatrix} O_1^{-1} & O_2^{-1} & O_3^{-1} \\ 1 & 1 & 1 \\ O_1^1 & O_2^1 & O_3^1 \end{vmatrix}. \tag{32}$$

Truncating the Taylor expansion in equation (31) to second order we get, after some algebraic manipulations,

$$O_1^1 O_3^{-1} \begin{vmatrix} O_1^{-1} & O_2^{-1} & O_3^{-1} \\ 1 & 1 & 1 \\ O_1^1 & O_2^1 & O_3^1 \end{vmatrix} = O_1^1 O_3^{-1} \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial}{\partial F_1} & \frac{\partial}{\partial F_2} & \frac{\partial}{\partial F_3} \\ \left(\frac{\partial}{\partial F_1}\right)^2 & \left(\frac{\partial}{\partial F_2}\right)^2 & \left(\frac{\partial}{\partial F_3}\right)^2 \end{vmatrix}. \tag{33}$$

Changing $\rho_{d;\mathcal{G}}(\gamma, \nu)$ in (30) to $\rho_d(f_1, f_2, f_3)$, we have

$$\rho_d(f_1, f_2, f_3) = \frac{1}{\sqrt{3}\pi\sigma_\gamma^2(m)} \exp\left[-\frac{2}{3\sigma_\gamma^2(m)} \left(\sum_{i=1}^{3} f_i^2 - \sum_{i<j}^{3} f_i f_j \right) \right]. \tag{34}$$

With this,

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial}{\partial F_1} & \frac{\partial}{\partial F_2} & \frac{\partial}{\partial F_3} \\ \left(\frac{\partial}{\partial F_1}\right)^2 & \left(\frac{\partial}{\partial F_2}\right)^2 & \left(\frac{\partial}{\partial F_3}\right)^2 \end{vmatrix} \frac{1}{\sqrt{3}\pi\sigma_\gamma^2(m)} \exp\left[-\frac{2}{3\sigma_\gamma^2(m)}\left(\sum_{i=1}^{3} F_i^2 - \sum_{i<j}^{3} F_i F_j\right)\right]$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ -\frac{2}{3\sigma_\gamma^2(m)}(2F_1 - F_2 - F_3) & -\frac{2}{3\sigma_\gamma^2(m)}(2F_2 - F_1 - F_3) & -\frac{2}{3\sigma_\gamma^2(m)}(2F_3 - F_1 - F_2) \\ \left\{\frac{4}{9\sigma_\gamma^4(m)}(2F_1 - F_2 - F_3)^2 - \frac{4}{3\sigma_\gamma^2(m)}\right\} & \left\{\frac{4}{9\sigma_\gamma^4(m)}(2F_2 - F_1 - F_3)^2 - \frac{4}{3\sigma_\gamma^2(m)}\right\} & \left\{\frac{4}{9\sigma_\gamma^4(m)}(2F_3 - F_1 - F_2)^2 - \frac{4}{3\sigma_\gamma^2(m)}\right\} \end{vmatrix}$$

$$\times \frac{1}{\sqrt{3}\pi\sigma_\gamma^2(m)} \exp\left[-\frac{2}{3\sigma_\gamma^2(m)}\left(\sum_{i=1}^{3} F_i^2 - \sum_{i<j}^{3} F_i F_j\right)\right]$$

$$= \frac{-2^3}{3^3\sigma_\gamma^6(m)} \begin{vmatrix} 0 & 0 & 1 \\ 3(F_1 - F_2) & 3(F_2 - F_3) & (2F_3 - F_1 - F_2) \\ 3(F_1 - F_2) \times (F_1 + F_2 - 2F_3) & 3(F_2 - F_3) \times (F_2 + F_3 - 2F_1) & (2F_3 - F_1 - F_2)^2 - 3\sigma_\gamma^2(m) \end{vmatrix}$$

$$\times \frac{1}{\sqrt{3}\pi\sigma_\gamma^2(m)} \exp\left[-\frac{2}{3\sigma_\gamma^2(m)}\left(\sum_{i=1}^{3} F_i^2 - \sum_{i<j}^{3} F_i F_j\right)\right]$$

$$= \frac{2^3}{\sqrt{3}\pi\sigma_\gamma^8(m)}(F_1 - F_2)(F_2 - F_3)(F_1 - F_3) \exp\left[-\frac{2}{3\sigma_\gamma^2(m)}\left(\sum_{i=1}^{3} F_i^2 - \sum_{i<j}^{3} F_i F_j\right)\right]. \tag{35}$$

Now applying equation (32) gives

$$O_1^1 O_3^{-1} \begin{vmatrix} O_1^{-1} & O_2^{-1} & O_3^{-1} \\ 1 & 1 & 1 \\ O_1^1 & O_2^1 & O_3^1 \end{vmatrix} \frac{1}{\sqrt{3}\pi\sigma_\gamma^2(m)} \exp\left[-\frac{2}{3\sigma_\gamma^2(m)}\left(\sum_{i=1}^{3} F_i^2 - \sum_{i<j}^{3} F_i F_j\right)\right]$$

$$= \frac{2^3}{\sqrt{3}\pi\sigma_\gamma^8(m)}(F_1 - F_2 + 1)(F_1 - F_3 + 2)(F_2 - F_3 + 1)$$

$$\times \exp -\frac{2}{3\sigma_\gamma^2(m)} \frac{\left[3\left\{(F_1+1)^2 + F_2^2 + (F_3 - 1)^2\right\} - \left(\sum_1^3 F_i\right)^2\right]}{2}. \tag{36}$$

Replacing $\lambda = F_1 - F_2$ and $\mu = F_2 - F_3$ in equation (36) we have finally

$$D^{(\lambda,\mu)}_{\{\eta\},\{m\}} = 3\binom{\mathcal{N} + m - 1}{m} \frac{2^3}{\sqrt{3}\pi\sigma_\gamma^8(m)}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2)$$

$$\times \exp\left(-\frac{2}{3\sigma_\gamma^2(m)}\{(\lambda + \mu + 3)(\lambda + \mu) - \lambda\mu + 3\}\right)$$

$$= 3\binom{\mathcal{N} + m - 1}{m} \frac{16}{\sqrt{3}\pi\sigma_\gamma^8(m)} d(\lambda, \mu) \exp\left(-\frac{2}{3\sigma_\gamma^2(m)}\{C_2(\lambda, \mu) + 3\}\right). \tag{37}$$

This formula was reported first, with incomplete discussion, by Kanestrom [8]. The factor 3 in equation (37) was not mentioned in [8] and the origin of this factor was discussed following equation (30). It is important to stress that we have identified the definition for $\sigma_\gamma^2(m)$ while Kanestrom treated it as a free parameter. Besides this we have derived a simple propagation formula for $\sigma_\gamma^2(m)$. Also we can extend the propagation method to higher order bivariate cumulants of ρ_d. With these we can add corrections to equation (37). Most important is the asymmetry in λ and μ seen in exact $D^{(\lambda\mu)}_{\{\eta\},\{m\}}$ while it is absent in the above formula. Let us now consider the third- and fourth-order bivariate moments and they will account for the asymmetry as we shall show now.

5. $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ with quadratic and cubic Casimir invariants of $SU(3)$

5.1. Result with third-order bivariate cumulants correction to the Gaussian form of $\rho_d(\gamma,\mu)$

By symmetry argument it is easy to see that $\langle\gamma^3\rangle^m = \langle(\widetilde{N}_z - \widetilde{N}_x)^3\rangle^m = 0$ and therefore the cumulants $k_{30}(m) = k_{03}(m) = 0$. Now we will consider the $k_{21}(m)$ cumulant. Firstly $k_{21}(m) = \langle\gamma^2\nu\rangle^m / \sigma_\gamma^3(m)$. Also from equation (A.4),

$$\begin{aligned}
\langle\gamma^2\nu\rangle^m &= \frac{m(\mathcal{N}+m)(\mathcal{N}+2m)}{\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)}\langle\langle\gamma^2\nu\rangle\rangle^1,\\
\langle\langle\gamma^2\nu\rangle\rangle^1 &= \langle(\widetilde{N}_z - \widetilde{N}_x)^2(\widetilde{N}_x - \widetilde{N}_y)\rangle^1\\
&= \langle\langle\widetilde{N}_z^3\rangle\rangle^1 - 3\langle\langle\widetilde{N}_z^2\widetilde{N}_x\rangle\rangle^1 + 2\langle\langle\widetilde{N}_z\widetilde{N}_x\widetilde{N}_y\rangle\rangle^1.
\end{aligned} \tag{38}$$

Here in the second equality we have used the symmetries of $\langle\widetilde{N}_i^3\rangle$ and $\langle\widetilde{N}_i^2\widetilde{N}_j\rangle$. The one particle averages in equation (38) follow by extending equations (27) and (28). Then,

$$\begin{aligned}
\langle\langle\widetilde{N}_z^3\rangle\rangle^1 &= \sum_{i=1}^{\eta+1}\left(\frac{2\eta}{3}+1-i\right)^3 (i)\\
&= \eta(\eta+1)(\eta+2)(\eta+3)(2\eta+3)/540,\\
\langle\langle\widetilde{N}_z^2\widetilde{N}_x\rangle\rangle^1 &= \sum_{i=0}^{\eta}\left(\frac{2\eta}{3}-i\right)^2\left[\sum_{j=0}^{i}\left(j-\frac{\eta}{3}\right)\right]\\
&= -\eta(\eta+1)(\eta+2)(\eta+3)(2\eta+3)/1080,\\
\langle\langle\widetilde{N}_z\widetilde{N}_x\widetilde{N}_y\rangle\rangle^1 &= \sum_{i=0}^{\eta}\left(\frac{2\eta}{3}-i\right)\left[\sum_{j=0}^{i}\left(j-\frac{\eta}{3}\right)\left(i-j-\frac{\eta}{3}\right)\right]\\
&= \eta(\eta+1)(\eta+2)(\eta+3)(2\eta+3)/540.
\end{aligned} \tag{39}$$

Combining equations (38), (39) and (29) we obtain the following formula for $k_{21}(m)$:

$$k_{21}(m) = \sqrt{\frac{3\mathcal{N}(\mathcal{N}+1)}{\eta(\eta+1)(\eta+2)(\eta+3)m(\mathcal{N}+m)}}\left[\frac{(2\eta+3)(\mathcal{N}+2m)}{5(\mathcal{N}+2)}\right]. \tag{40}$$

For example for $\eta = 4$ (then $\mathcal{N} = 15$) and $m = 20$, equation (40) gives $k_{21}(m) = (\sqrt{3/2})(121/595)$ and we have verified this by numerically generating $d(\gamma,\nu)$. It is easy to see that $k_{12}(m) = -k_{21}(m)$. In many examples it is found that $k_{21} \sim 0.25$ and therefore the third-order cumulants generate important corrections to equation (37). Employing the bivariate ED (Edgeworth) expansion given in appendix B, which has well understood convergence properties (see [20, 21]), and using the results $k_{30} = k_{03}$ and $k_{21} = -k_{12}$, we have the following expression for ρ_d with first correction:

$$\rho_{d;ED}(\gamma,\nu) = \rho_{d;\mathcal{G}}(\gamma,\nu)\left[1 + \frac{k_{21}}{2}\{He_{21}(\boldsymbol{\gamma},\boldsymbol{\nu}) - He_{12}(\boldsymbol{\gamma},\boldsymbol{\nu})\}\right]. \quad (41)$$

The $\boldsymbol{\gamma}$ and $\boldsymbol{\nu}$ are standard variables and $He_{21}(x, y)$ for standard x and y is

$$He_{21}(x,y) = \frac{(x-\zeta y)^2(y-\zeta x)}{(1-\zeta^2)^3} + 2\zeta\frac{x-\zeta y}{(1-\zeta^2)^2} - \frac{y-\zeta x}{(1-\zeta^2)^2}. \quad (42)$$

Similarly $He_{12}(x, y)$ is defined with $x \leftrightarrow y$. Using the fact that for the variables (γ, ν), the correlation coefficient is $-\frac{1}{2}$ will simplify equation (41) to give

$$\rho_{d;ED}(\gamma,\nu) = \rho_{d;\mathcal{G}}(\gamma,\nu)\left[1 + \frac{4k_{21}}{27\sigma_\gamma^3(m)}(2\gamma+\nu)(2\nu+\gamma)(\gamma-\nu)\right]. \quad (43)$$

Note the appearance of the term that has a structure close to that of $C_3(\lambda, \mu)$ and this is in line with the expectation that C_3 generates the asymmetry in the D-function with respect to λ and μ. Now applying equation (33) and carrying out the simplifications indeed generate a term containing C_3. The resulting remarkably simple and easy to understand formula with k_{21} and k_{12} corrections for the D function is

$$D^{(\lambda\mu)}_{\{\eta\},\{m\}} = 3\binom{\mathcal{N}+m-1}{m}\frac{16}{\sqrt{3}\pi\sigma_\gamma^8(m)}d(\lambda,\mu) \times \exp\left(-\frac{2}{3\sigma_\gamma^2(m)}\{C_2(\lambda,\mu)+3\}\right)\left[1 + \frac{4k_{21}(m)}{3\sigma_\gamma^3(m)}C_3(\lambda,\mu)\right]. \quad (44)$$

5.2. *Result with fourth-order bivariate cumulants correction to the Gaussian form of $\rho_d(\gamma, \mu)$*

Going beyond the first correction discussed in the previous section, using the ED expansion given by equation (B.4) we can include also the second-order corrections. For this we need the cumulants k_{rs} or the central moments $\mathcal{M}_{rs} = \langle\gamma^r\nu^s\rangle^m = \langle(\widetilde{N}_z - \widetilde{N}_x)^r(\widetilde{N}_x - \widetilde{N}_y)^s\rangle^m$ with $r + s = 4$. Firstly, by symmetry arguments it is easy to see that $k_{40} = k_{04}$ and $k_{31} = k_{13}$. For example

$$\sigma_\gamma^4(m)k_{40}(m) = 2\langle\widetilde{N}_z^4\rangle^m - 8\langle\widetilde{N}_z^3\widetilde{N}_x\rangle^m + 6\langle\widetilde{N}_z^2\widetilde{N}_x^2\rangle^m - 3\sigma_\gamma^4(m).$$

Similarly writing the bivariate cumulants k_{22} and k_{31} in terms of the bivariate moments using equation (B.6), substituting $\zeta = -1/2$ and using the symmetries of the traces involving $\widetilde{N}_i$ operators, it is seen that $k_{22} = k_{40}/2$ and $k_{13} = -k_{40}/2$. Therefore all we need is the propagation formula for $k_{40}(m)$. To this end we use equation (A.5) and then we need

$$\langle\langle(\widetilde{N}_z - \widetilde{N}_x)^4\rangle\rangle^1 = 2\langle\langle\widetilde{N}_z^4\rangle\rangle^1 - 8\langle\langle\widetilde{N}_z^3\widetilde{N}_x\rangle\rangle^1 + 6\langle\langle\widetilde{N}_z^2\widetilde{N}_x^2\rangle\rangle^1.$$

These one particle traces are evaluated using the procedure followed in the previous sections and then

$$\langle\langle(\widetilde{N}_z - \widetilde{N}_x)^4\rangle\rangle^1 = \tfrac{1}{60}\eta(\eta+1)(\eta+2)(\eta+3)(2\eta^2+6\eta-3). \quad (45)$$

Now the propagation equation for $k_{40}(m)$ is

$$k_{40}(m) = \left[\frac{\mathcal{N}(\mathcal{N}+1)}{m(\mathcal{N}+m)} + 6\frac{(m-1)(\mathcal{N}+m+1)\mathcal{N}(\mathcal{N}+1)}{m(\mathcal{N}+m)(\mathcal{N}+2)(\mathcal{N}+3)}\right]\frac{12(2\eta^2+6\eta-3)}{5\eta(\eta+1)(\eta+2)(\eta+3)} + 3\left[\frac{(m-1)(\mathcal{N}+m+1)\mathcal{N}(\mathcal{N}+1)}{m(\mathcal{N}+m)(\mathcal{N}+2)(\mathcal{N}+3)} - 1\right]. \quad (46)$$

Thus we can calculate k_{40} for any m. Equation (B.4) and the symmetries of k_{rs} will give the second-order correction to be (we will denote this by X without the Gaussian pre-factor),

$$X = k_{40}\big[\tfrac{1}{24}\{He_{40}(\gamma,\nu) + He_{04}(\gamma,\nu)\} - \tfrac{1}{12}\{He_{31}(\gamma,\nu) + He_{13}(\gamma,\nu)\} + \tfrac{1}{8}He_{22}(\gamma,\nu)\big] + k_{21}^2\big[\tfrac{1}{8}\{He_{42}(\gamma,\nu) + He_{24}(\gamma,\nu)\} - \tfrac{1}{4}He_{33}(\gamma,\nu)\big]. \tag{47}$$

Here we used the result $k_{30} = k_{03} = 0$ and the Hermite polynomials $He(--)$ followed from equation (B.5). Substituting $\zeta = -1/2$ and simplifying the Hermite polynomials will give the following pleasing result:

$$\begin{gathered} X = \tfrac{1}{27}k_{40}[2X_2(X_2-6)+9] + \tfrac{4}{729}k_{21}^2\big[2X_3^2 - 54X_2^2 + 81(2X_2-1)\big], \\ X_2 = \gamma^2 + \gamma\nu + \nu^2, \qquad X_3 = (\gamma-\nu)(2\gamma+\nu)(\gamma+2\nu). \end{gathered} \tag{48}$$

Note the appearance of the terms that have a structure close to that of $C_2(\lambda,\mu)$ and $C_3(\lambda,\mu)$ and this is in line with the expectation that the D-function should be a function of only these two invariants. Now applying equation (33) and carrying out the simplifications we obtain the D-function including second-order corrections,

$$\begin{aligned} D^{(\lambda\mu)}_{\{\eta\},\{m\}} &= 3\binom{\mathcal{N}+m-1}{m}\frac{16}{\sqrt{3}\pi\sigma_\gamma^8(m)}d(\lambda,\mu) \\ &\quad\times \exp\left(-\frac{2}{3\sigma_\gamma^2(m)}\{C_2(\lambda,\mu)+3\}\right)\left[1 + \frac{4k_{21}(m)}{3\sigma_\gamma^3(m)}C_3(\lambda,\mu)\right. \\ &\quad+\left\{\frac{8k_{21}^2(m)}{9\sigma_\gamma^6(m)}C_3^2(\lambda,\mu) - \frac{2\left(4k_{21}^2(m)-k_{40}(m)\right)}{27\sigma_\gamma^4(m)}[C_2(\lambda,\mu)+3]^2\right. \\ &\quad\left.\left.+\frac{10\left(2k_{21}^2(m)-k_{40}(m)\right)}{9\sigma_\gamma^2(m)}[C_2(\lambda,\mu)+3] + \left(-\frac{40}{9}k_{21}^2(m) + \frac{10}{3}k_{40}\right)\right\}\right]. \end{aligned} \tag{49}$$

Equation (49) is the main result of this paper and it is expected to work well for $\lambda,\mu \lesssim 3\sigma_\gamma(m)$.

5.3. Numerical tests of the statistical formula

In order to test the statistical formulae given by equations (37), (44) and (49), we have carried out numerical calculations for $\eta = 4, m = 20$ and $40, \eta = 5, m = 24$ and $\eta = 6, m = 18$. Firstly, it is seen from exact results (obtained using the methods described in section 2) that the multiplicities for (λ,μ) and (μ,λ) irreps are in general quite different. For example for $[(0,30),(30,0)]$, $[(2,32),(32,2)]$ and $[(4,34),(34,4)]$ irreps they are (64 124, 127 293), (133 652, 329 111) and (141 146, 447 569) respectively for $\eta = 5, m = 24$. This asymmetry in the multiplicities will not follow from the bivariate Gaussian approximation given by equation (37) which contains only the $C_2(\lambda,\mu)$ term that is symmetric in λ and μ. Thus $C_3(\lambda,\mu)$, which is asymmetric in λ and μ, is needed for proper description of the multiplicities. We have seen in the previous section that the introduction of Edgeworth corrections to the bivariate Gaussian naturally introduces terms with $C_3(\lambda,\mu)$. Using the formulae given by equations (29), (40) and (46) numerical values for the variance $\sigma_\nu^2(m)$ and the cumulants $k_{21}(m)$ and $k_{40}(m)$ for different values of η and m are calculated and for some example results are shown in table 1; for completeness the total dimension $d(\mathcal{N},m)$ is also given. It is clearly seen from the examples in table 1 that $k_{21} \sim 0.25$ and $k_{40} \sim 0.05$. The variances and the cumulants in table 1 are used to construct $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ and for the example of $\eta = 4$, $m = 40$, the results are shown in figure 1 as 3D histograms. Here the exact results are

Table 1. Marginal variance $\sigma_\gamma^2(m)$ and the bivariate cumulants $k_{21}(m)$ and $k_{40}(m)$ for $\rho_d(\gamma, \nu)$.

η	m	$d(\mathcal{N}, m)$	$\sigma_\gamma^2(m)$	$k_{21}(m)$	$k_{40}(m)$
4	20	1391 975 640	$\frac{1225}{6}$	$\frac{121\sqrt{3/2}}{595}$	$\frac{141}{2975}$
4	40	3245 372 870 670	$\frac{1925}{3}$	$\frac{19\sqrt{33/7}}{170}$	$\frac{1956}{32725}$
5	24	1761 039 350 070	$\frac{3600}{11}$	$\frac{13\sqrt{11}}{200}$	$\frac{23}{480}$
6	18	1715 884 494 940	$\frac{7452}{29}$	$\frac{8\sqrt{29/23}}{45}$	$\frac{982}{32085}$

compared with Edgeworth corrections to second order, i.e. equation (49). The histogram plots are constructed by first binning the numerically obtained for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ with bin size 4 for both λ and μ. The numerical value of the D-function in each bin in then divided by the area 16. With the introduction of second-order Edgeworth corrections we see that there is good agreement between the 3D histogram plots. However to bring out finer differences, in figures 2–4 we show D versus μ for various λ values and the results are for $(\eta = 4, m = 40)$, $(\eta = 5, m = 24)$ and $(\eta = 6, m = 18)$ respectively. From these figures it is clearly seen that in all the cases the bivariate Gaussian results given by equation (37) deviate from the exact results except for very small values of λ and μ. When the first correction (from k_{21}) is added to the Gaussian approximation, the agreement between the exact and the approximate results improves significantly. For $\eta = 4$ and $m = 40$, there is good agreement for $\lambda, \mu \lesssim 50$. From the numerical values listed in table 1 we see that for this example $\sigma_\gamma(m) \sim 25$. In general, the values of λ and μ up to which there is agreement between the exact and approximate densities is $\sim 2\sigma_\gamma(m)$; for the $(\eta = 5, m = 24)$ and $(\eta = 6, m = 18)$ examples, $\sigma_\gamma(m)$ are ~ 18 and ~ 16, respectively. Addition of the second-order corrections to the Gaussian approximation results in good agreement between the exact and approximate densities. From figure 2 it is seen that for $(\eta = 4, m = 40)$, the agreement is good for up to $\lambda, \mu \lesssim 3\sigma_\gamma(m)$. Similar conclusion can be drawn from figures 3 and 4 for $(\eta = 5, m = 24)$ and $(\eta = 6, m = 18)$, respectively. It is important to add that for the three $\eta = 5, m = 24$ examples mentioned in the beginning of this section for the multiplicities of $[(0, 30), (30, 0)]$, $[(2, 32), (32, 2)]$ and $[(4, 34), (34, 4)]$, equation (49) gives values $(64\,953, 124\,740)$, $(138\,789, 322\,282)$ and $(150\,894, 434\,969)$ respectively. These are in close agreement with exact values. Thus we can conclude that the analytical expression given by equation (49) is a good asymptotic expression for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$.

6. Conclusions

A statistical law for the multiplicities $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ of the $SU(3)$ irreps (λ, μ) in the reduction of a totally symmetric irreducible representation $\{m\}$ of $U(\mathcal{N})$, $\mathcal{N} = (\eta + 1)(\eta + 2)/2$ is derived in terms of the quadratic and cubic invariants of $SU(3)$ for the first time in this paper. To this end, the first three terms of an asymptotic expansion, based on the ED expansion well known in statistics, are determined from first principles. They are given in equations (37), (44) and (49). In addition, simple formulae in terms of m and η, for all the parameters in the expansion are derived. They are given in equations (29), (40) and (46). Numerical tests with large m and $\eta = 4, 5$ and 6 show good agreement between exact results and the statistical formula with second-order corrections. Although we have restricted ourselves to boson systems, it is possible to extend the results of the paper to fermion systems, i.e. antisymmetric irreps of $U(\mathcal{N})$ and also for a general $U(\mathcal{N})$ irrep (for example, they appear with the spin–isospin $SU(4)$ symmetry or just with spin in nuclei [9, 12]). Toward this end we need to derive new

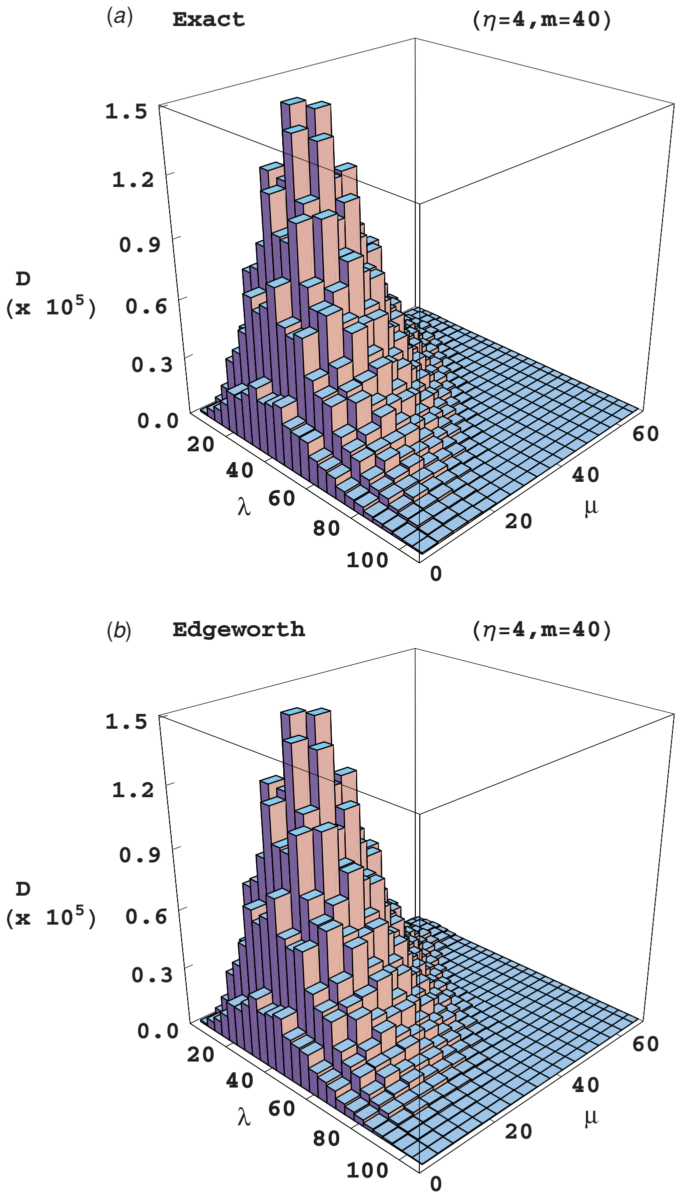

Figure 1. (*a*) 3D histogram for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ from exact calculation. (*b*) 3D histogram for $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ calculated using the analytical result for bivariate Gaussian approximation with Edgeworth corrections up to second order as given by equation (49).

propagation formulae for the bivariate cumulants k_{rs} with $r + s \leqslant 4$ and this will be addressed elsewhere.

The study carried out in this paper represents a step forward in the subject of 'statistical group theory' [3, 22]. We hope that the results reported here, obtained nearly 30 years after Kanestrom's [8] first attempt, will prompt further studies in statistical group theory and we expect more applications of this topic in physics in future.

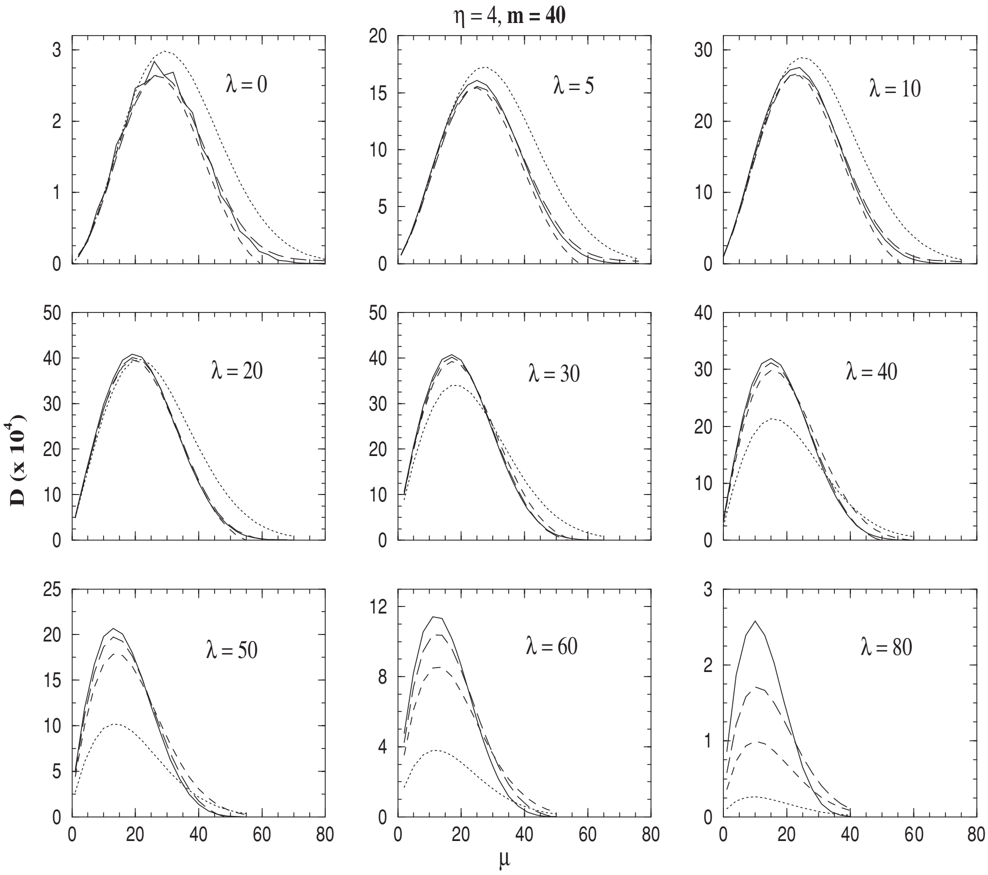

Figure 2. Plot of $D^{(\lambda,\mu)}_{\{\eta\},\{m\}}$ versus μ for different values of λ with $\eta = 4$ and $m = 40$. Here the solid curve corresponds to the exact result, the dotted curve corresponds to the bivariate Gaussian approximation, i.e. equation (37), the dashed curve corresponds to the bivariate Gaussian with first-order Edgeworth correction, i.e. equation (44) and the long dashed curve corresponds to the bivariate Gaussian approximation with second-order Edgeworth corrections, i.e. equation (49).

Appendix A.

Given a k-body operator $\mathcal{O}(k)$, its m particle average is

$$\langle \mathcal{O}(k)\rangle^m = \{d(m)\}^{-1}\langle\langle \mathcal{O}(k)\rangle\rangle^m = \{d(m)\}^{-1}\sum_\alpha \langle m\alpha|\mathcal{O}(k)|m\alpha\rangle$$
$$= \binom{m}{k}\langle \mathcal{O}(k)\rangle^k . \tag{A.1}$$

In equation (A.1), $d(m)$ is the dimension of the m particle space for the particles occupying $\mathcal{N}$ sp states. Note that $\binom{m}{k}$ has the correct particle rank, $\langle \mathcal{O}(k)\rangle^m = 0$ for $m < k$ and $\langle \mathcal{O}(k)\rangle^m = \langle \mathcal{O}(k)\rangle^k$ for $m = k$. Let us consider the averages of powers of a one-body Hamiltonian $h(1)$ over a m boson space. We will consider only diagonal sp energies ϵ_i, i.e. $h(1) = \sum_{i=1}^{\mathcal{N}} \epsilon_i \hat{n}_i$ where $|i\rangle$ are sp states and $\hat{n}_i$ are number operators for the states i. Decomposing $h^r(1)$ into $(0 + 1 + 2 + \cdots + r)$-body operators and applying equation (A.1) to

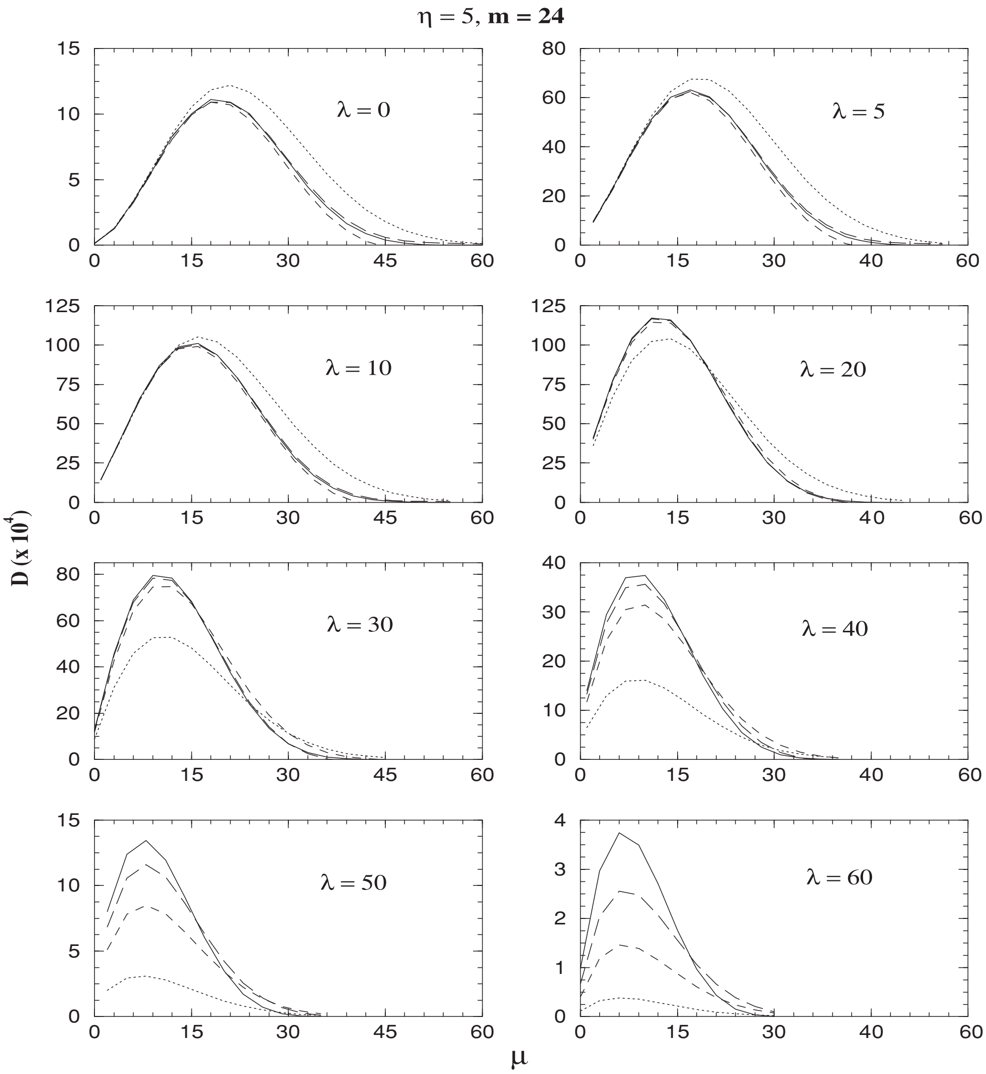

Figure 3. Same as figure 2 but for $\eta = 5$ and $m = 24$.

each piece separately, trace propagation equations for $\langle h^r(1)\rangle^m$ can be derived with inputs containing ϵ_is explicitly. For $r = 1$–4 the propagation equations are [19]

$$\langle h(1)\rangle^m = m\overline{\epsilon}, \qquad \overline{\epsilon} = \{\mathcal{N}\}^{-1}\sum_{i=1}^{\mathcal{N}} \epsilon_i. \tag{A.2}$$

$$\langle \tilde{h}^2(1)\rangle^m = \frac{m(\mathcal{N}+m)}{\mathcal{N}(\mathcal{N}+1)}\sum_{i=1}^{\mathcal{N}} \tilde{\epsilon}_i^2, \qquad \tilde{\epsilon}_i = \epsilon_i - \overline{\epsilon}, \qquad \tilde{h}(1) = \sum_{i=1}^{\mathcal{N}} \tilde{\epsilon}_i \hat{n}_i. \tag{A.3}$$

$$\langle \tilde{h}^3(1)\rangle^m = \frac{m(\mathcal{N}+m)(\mathcal{N}+2m)}{\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)}\sum_{i=1}^{\mathcal{N}} \tilde{\epsilon}_i^3. \tag{A.4}$$

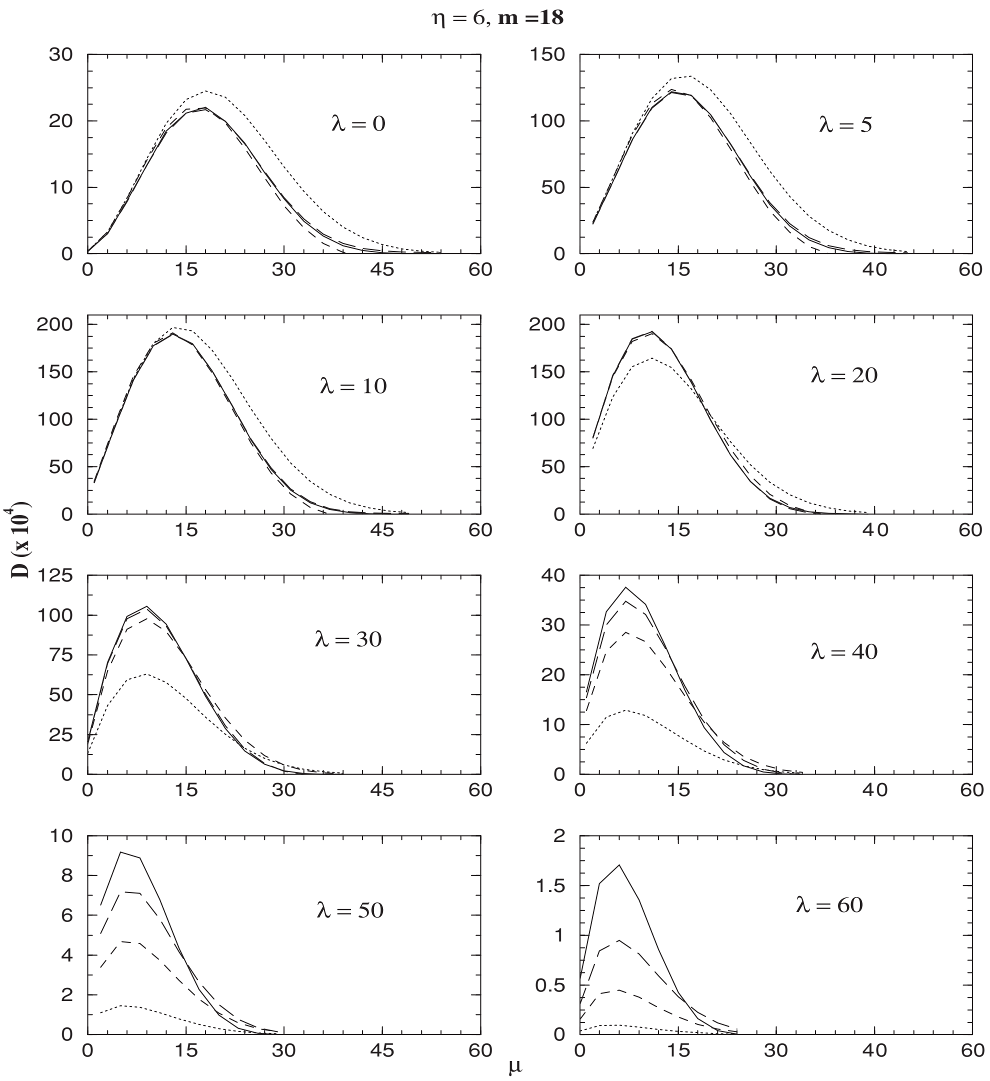

Figure 4. Same as figure 2 but for $\eta = 6$ and $m = 18$.

$$\langle \tilde{h}^4(1)\rangle^m = \frac{m(\mathcal{N}+m)}{\mathcal{N}(\mathcal{N}+1)}\sum_{i=1}^{\mathcal{N}}\tilde{\epsilon}_i^4 + \frac{m(m-1)(\mathcal{N}+m)(\mathcal{N}+m+1)}{\mathcal{N}(\mathcal{N}+1)(\mathcal{N}+2)(\mathcal{N}+3)}\left[3\left(\sum_{i=1}^{\mathcal{N}}\tilde{\epsilon}_i^2\right)^2 + 6\sum_{i=1}^{\mathcal{N}}\tilde{\epsilon}_i^4\right]. \tag{A.5}$$

Note that $\langle h(1)\rangle^m$ gives the centroid of $\rho^m(E) = \langle\delta(h(1)-E)\rangle^m$, the m-particle density of states. Similarly $\sigma(m) = \sqrt{\langle\tilde{h}^2(1)\rangle^m}$ is the width of $\rho(E)$. For symmetric ϵ_is clearly $\langle\tilde{h}^3(1)\rangle^m = 0$ and the shape of $\rho(E)$ is then largely decided by the excess parameter $\gamma_2(m) = [\langle\tilde{h}^4(1)\rangle^m/\sigma^4(m)] - 3$. For non-singular spectra (i.e. for $h(1)$ with $|\gamma_2(1)| \lesssim 1$), it is seen (by applying equations (A.3) and (A.5)) that $\gamma_2(m) \sim 0$ in the dense limit, i.e. in the limit $m \to \infty, \mathcal{N} \to \infty$ and $m/\mathcal{N} \to \infty$. Therefore for dense non-interacting boson systems there is central limit theorem (CLT) action giving Gaussian density of states [19]. Now we will apply this result and equations (A.2)–(A.5) to derive statistical laws for $D(m, L)$.

Let us consider a system of bosons carrying angular momenta $(\ell_1, \ell_2, \ldots, \ell_r)$. Then the sp spectrum for the ℓ_z operator consists of k $[k = 2\{\max(\ell_1, \ell_2, \ldots, \ell_r)\} + 1]$ number of ℓ_z eigenvalues m_{z_i}, each with degeneracy d_i. For example for $(sdg)^m$ system, $m_{z_i} = -4, -3, -2, -1, 0, 1, 2, 3, 4$ with $d_i = 1, 1, 2, 2, 3, 2, 2, 1, 1$ respectively. Now distributing a given number of bosons m in the ℓ_z orbits in all possible ways, the degeneracy $\mathcal{D}(m, M)$ for a given total L_z eigenvalue M is easy to obtain. Then the simple difference formula

$$D(m, L) = \mathcal{D}(m, M = L) - \mathcal{D}(m, M = L + 1) \tag{A.6}$$

gives the fixed-L dimension $D(m, L)$ or equivalently the multiplicity of the irrep $[L]$ of $SO(3)$ in the reduction of the irrep $\{m\}$ of $U(\mathcal{N})$. Recognizing that $\mathcal{D}(m, M)/d(m)$ is the same as the density of L_z eigenvalues in m boson space, i.e. $\rho^m(M) = \{d(m)\}^{-1}\langle\langle\delta(L_z - M)\rangle\rangle^m$, statistical laws for $\mathcal{D}(m, M)$ and hence, via equation (A.6), for $D(m, L)$ can be obtained. As m_{z_i} are additive, ρ^m will be a m-fold convolution of ρ^1 and hence there is CLT action in generating $\rho^m(M)$. Then

$$\begin{aligned} \mathcal{D}(m, M) &= \frac{d(m)}{\sqrt{2\pi}\sigma_L(m)} \exp -\left(\frac{M^2}{2\sigma_L^2(m)}\right), \\ \sigma_L^2 = \langle L_z^2\rangle^m &= \frac{m(\mathcal{N} + m)}{\mathcal{N}(\mathcal{N} + 1)} \sum_{i=1}^{k} m_{z_i}^2 d_i. \end{aligned} \tag{A.7}$$

Equation (A.7) is derived using equation (A.3) by noting that $\langle L_z\rangle^m = 0$. Using equation (A.7), the statistical law for $D(m, L)$ is

$$\begin{aligned} D(m, L) &= \mathcal{D}(m, M = L) - \mathcal{D}(m, M = L + 1) \\ &\simeq -\frac{\partial}{\partial L}\mathcal{D}\left(m, M = L + \frac{1}{2}\right) \\ &\xrightarrow{CLT} \binom{\mathcal{N} + m - 1}{m} \frac{(2L + 1)}{\sqrt{8\pi\sigma_L^3(m)}} \exp -\frac{\left(L + \frac{1}{2}\right)^2}{2\sigma_L^2(m)}. \end{aligned} \tag{A.8}$$

One can go beyond the Gaussian approximation in (A.7) and improve equation (A.8) by adding Edgeworth corrections; see [1, 20] for univariate Edgeworth expansion. In addition to (A.8), statistical laws for $D(L)$ with fixed particle number in each ℓ-orbit or groups of such orbits (for example sd and pf in $sdpf$ IBM) can be derived. As an important by product this gives not only fixed-L but also fixed-L and parity dimensions; see [1].

Appendix B.

Given the bivariate Gaussian, in terms of the standardized variables $\widehat{x}$ and $\widehat{y}$,

$$\eta_{\mathcal{G}}(\widehat{x}, \widehat{y}) = \frac{1}{2\pi\sqrt{(1 - \zeta^2)}} \exp\left\{-\frac{\widehat{x}^2 - 2\zeta\widehat{x}\widehat{y} + \widehat{y}^2}{2(1 - \zeta^2)}\right\} \tag{B.1}$$

the bivariate ED expansion for any bivariate distribution $\eta(\widehat{x}, \widehat{y})$ follows from

$$\eta(\widehat{x}, \widehat{y}) = \exp\left\{\sum_{r+s\geqslant 3} (-1)^{r+s} \frac{k_{rs}}{r!s!} \frac{\partial^r}{\partial\widehat{x}^r} \frac{\partial^s}{\partial\widehat{y}^s}\right\} \eta_{\mathcal{G}}(\widehat{x}, \widehat{y}). \tag{B.2}$$

Assuming that the bivariate reduced cumulants k_{r+s} behave as $k_{r_s} \propto \Upsilon^{-(r+s-2)/2}$ where Υ is a system parameter, and collecting in the expansion of equation (B.2) all the terms that behave as $\Upsilon^{-P/2}$, $P = 1, 2, \ldots$, we obtain the bivariate ED expansion. Then [20, 21],

$$\eta_{ED}(\widehat{x},\widehat{y}) = \eta_{\mathcal{G}}(\widehat{x},\widehat{y}) \left\{ 1 + \sum_{P=1}^{\infty} \left[\sum_{[P]} \sum_{\{\prod_{i=1}^{m}(p_i q_i)^{\pi_i}\}} \left[\prod_{i=1}^{m} \left\{ \frac{k_{p_i q_i}}{p_i! q_i!} \right\}^{\pi_i} \frac{1}{\pi_i!} \right] He_{\sum p_i \pi_i, \sum q_i \pi_i}(\widehat{x},\widehat{y}) \right] \right\}. \tag{B.3}$$

In equation (B.3), $[P]$ are the partitions of the integer P and the corresponding bipartitions $\prod_{i=1}^{m}(p_i q_i)^{\pi_i}$ are defined as follows. With $[P] = [P_1, P_2, \ldots, P_\ell]$, $P_1 \geqslant P_2 \geqslant \cdots \geqslant P_\ell > 0$, generate all possible $[(p_1q_1)(p_2q_2)\cdots(p_\ell q_\ell)]$ such that $p_i + q_i = P_i + 2$ and $p_i, q_i \geqslant 0$. If $(p_i q_i)$ is repeated π_i times, the bipartition is written as $\prod_{i=1}^{m}(p_i q_i)^{\pi_i} = [(p_1q_1)^{\pi_1}(p_2q_2)^{\pi_2}\cdots(p_m q_m)^{\pi_m}]$. Note that $\sum_{i=1}^{m}\pi_i = \ell$. The bivariate ED expansion to order $P = 2$ is

$$\begin{aligned}
\eta_{biv-ED}(\widehat{x},\widehat{y}) = \Bigg\{ 1 &+ \left(\frac{k_{30}}{6} He_{30}(\widehat{x},\widehat{y}) + \frac{k_{21}}{2} He_{21}(\widehat{x},\widehat{y}) \right. \\
&+ \left. \frac{k_{12}}{2} He_{12}(\widehat{x},\widehat{y}) + \frac{k_{03}}{6} He_{03}(\widehat{x},\widehat{y}) \right) \\
&+ \left(\left\{ \frac{k_{40}}{24} He_{40}(\widehat{x},\widehat{y}) + \frac{k_{31}}{6} He_{31}(\widehat{x},\widehat{y}) \right. \right. \\
&+ \left. \frac{k_{22}}{4} He_{22}(\widehat{x},\widehat{y}) + \frac{k_{13}}{6} He_{13}(\widehat{x},\widehat{y}) + \frac{k_{04}}{24} He_{04}(\widehat{x},\widehat{y}) \right\} \\
&+ \left\{ \frac{k_{30}^2}{72} He_{60}(\widehat{x},\widehat{y}) + \frac{k_{30}k_{21}}{12} He_{51}(\widehat{x},\widehat{y}) \right. \\
&+ \left[\frac{k_{21}^2}{8} + \frac{k_{30}k_{12}}{12} \right] He_{42}(\widehat{x},\widehat{y}) \\
&+ \left[\frac{k_{30}k_{03}}{36} + \frac{k_{12}k_{21}}{4} \right] He_{33}(\widehat{x},\widehat{y}) \\
&+ \left[\frac{k_{12}^2}{8} + \frac{k_{21}k_{03}}{12} \right] He_{24}(\widehat{x},\widehat{y}) + \frac{k_{12}k_{03}}{12} He_{15}(\widehat{x},\widehat{y}) \\
&+ \left. \left. \frac{k_{03}^2}{72} He_{06}(\widehat{x},\widehat{y}) \right\} \right) \Bigg\} \eta_{\mathcal{G}}(\widehat{x},\widehat{y}).
\end{aligned} \tag{B.4}$$

The bivariate Hermite polynomials $He_{m_1 m_2}(\widehat{x},\widehat{y})$ in equation (B.4) are generated by

$$He_{m_1 m_2}(\widehat{x},\widehat{y}) = [\eta_{\mathcal{G}}(\widehat{x},\widehat{y})]^{-1}(-1)^{m_1+m_2} \frac{\partial^{m_1}}{\partial \widehat{x}^{m_1}} \frac{\partial^{m_2}}{\partial \widehat{y}^{m_2}} \eta_{\mathcal{G}}(\widehat{x},\widehat{y}). \tag{B.5}$$

Note that $He_{m_1 m_2}(\widehat{x},\widehat{y}) = He_{m_2 m_1}(\widehat{y},\widehat{x})$. For completeness, we give here the bivariate cumulants K_{rs} for $r + s \leqslant 4$ in terms of the central moments $\mathcal{M}_{r'+s'}$ [20],

$$\begin{aligned}
K_{30} &= \mathcal{M}_{30} \\
K_{21} &= \mathcal{M}_{21} \\
K_{40} &= \mathcal{M}_{40} - 3\mathcal{M}_{20}^2 \\
K_{31} &= \mathcal{M}_{31} - 3\mathcal{M}_{20}\mathcal{M}_{11} \\
K_{22} &= \mathcal{M}_{22} - \mathcal{M}_{20}\mathcal{M}_{02} - 2\mathcal{M}_{11}^2
\end{aligned} \tag{B.6}$$

Note that $K_{rs} \to K_{sr}$ with $\mathcal{M}_{r's'} \to \mathcal{M}_{s'r'}$. Similarly $K_{20} = \sigma_{20}^2 = \mathcal{M}_{20}$ and $K_{02} = \sigma_{02}^2 = \mathcal{M}_{02}$. The reduced cumulants $k_{rs} = K_{rs}/[\{K_{20}\}^{r/2}\{K_{02}\}^{s/2}]$.

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